INTERPOLATION WITH UNEQUAL INTERVALS:

DIVIDED DIFFERENCES:

DIVIDED DIFFERENCE TABLE:

Argument (x)	Entry $f(x)$	First D.D $\Delta f(x)$	Second D.D $\Delta^2 f(x)$	Third D.D $\Delta^3 f(x)$
x_0	$f(x_0)$			
		$\frac{f(x_1)-f(x_0)}{x_1-x_0} = f(x_0,x_1)$		
x_1	$f(x_1)$		$f(x_0,x_1,x_2)$	
		$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f(x_1,x_2)$		$f(x_0,x_1,x_2,x_3)$
x_2	$f(x_2)$		$f(x_1,x_2,x_3)$	
		$\frac{f(x_3)-f(x_2)}{x_3-x_2} = f(x_2,x_3)$		
x_3	$f(x_3)$			

Properties of divided differences:

1. The divided differences are symmetrical in all their arguments.

2. The operator Δ is linear

3. The n^{th} divided differences of a polynomial of n^{th} degree are constant.

Problems:

1 If $f(x) = \frac{1}{x}$ find the divided difference $f(a,b,c,d)$ or $\Delta^{3}_{bcd}\left(\frac{1}{a}\right)$.

Solution:-

$$f(x) = \frac{1}{x} \Rightarrow f(a) = \frac{1}{a}$$

$$f(a,b) = \frac{f(b) - f(a)}{b-a} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = \frac{\left(\frac{a-b}{ba}\right)}{b-a}$$

$$= \frac{\left(\frac{\cancel{a-b}}{ab}\right)}{\cancel{b-a}} = \frac{-1}{ab}.$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a}$$

$$= \frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a} = \frac{\left[\frac{\cancel{-a+c}}{abc}\right]}{\cancel{c-a}} = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = \frac{\left(\frac{\cancel{a-d}}{abcd}\right)}{\cancel{d-a}} = -\frac{1}{abcd}.$$

$$\therefore f(a,b,c,d) = \Delta^{3}_{bcd}\left(\frac{1}{a}\right) = -\frac{1}{abcd}.$$

(2)

If $f(x) = \frac{1}{x^2}$ find the divided differences $f(a,b)$ and $f(a,b,c)$.

<u>Solution:</u>

Here $f(x) = \frac{1}{x^2}$

$$f(a) = \frac{1}{a^2} \; ; \; f(b) = \frac{1}{b^2} \text{ and } f(c) = \frac{1}{c^2}.$$

$$\therefore f(a,b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a} = \frac{\left(\frac{a^2-b^2}{a^2b^2}\right)}{b-a}$$

$$= \frac{\left[\frac{(a+b)\cancel{(a-b)}}{a^2b^2}\right]}{-\cancel{(a-b)}} = -\left(\frac{a+b}{a^2b^2}\right)$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a}$$

$$= \frac{-\left(\frac{b+c}{b^2c^2}\right) + \left(\frac{a+b}{a^2b^2}\right)}{c-a} = \frac{\left[\frac{-a^2(b+c)+c^2(a+b)}{a^2b^2c^2}\right]}{c-a}$$

$$= \frac{\left[\frac{-a^2b - a^2c + c^2a + c^2b}{a^2b^2c^2}\right]}{c-a}$$

$$= \frac{1}{c-a}\left[\frac{ac[c-a] + b[c^2-a^2]}{a^2b^2c^2}\right] = \frac{\cancel{c-a}}{\cancel{c-a}}\left[\frac{ac+b(c+a)}{a^2b^2c^2}\right]$$

$$= \frac{ac+bc+ba}{a^2b^2c^2}$$

//.

3. Find the third divided differences with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

Solution :

x : 2 4 9 10

$f(x)$: 4 56 711 980

Divided difference table :

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
2	4	$\frac{56-4}{4-2} = 26$		
			$\frac{131-26}{9-2} = 15$	
4	56	$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
			$\frac{269-131}{10-4} = 23$	
9	711	$\frac{980-711}{10-9} = 269$		
10	980			

Newton's divided difference formula (or)
Newton's interpolation formula for unequal intervals:

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$

$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \dots\dots$$

$$+ (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0,x_1\dots x_n).$$

4. Given the data

x :	0	1	2	5
$f(x)$:	2	3	12	147

Find the polynomial equation. Hence find $f(3)$.

Solution :-

The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2 $f(x_0)$	$f(x_0,x_1)$ $\frac{3-2}{1-0}=1$	$f(x_0,x_1,x_2)$ $\frac{9-1}{2-0}=4$	$f(x_0,x_1,x_2,x_3)$ $\frac{9-4}{5-0}=1$
1	3	$\frac{12-3}{2-1}=9$	$\frac{45-9}{5-1}=9$	
2	12	$\frac{147-12}{5-2}=45$		
5	147			

Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \dots \rightarrow (1)$$

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$.

$f(x_0) = 2$, $f(x_0,x_1) = 1$, $f(x_0,x_1,x_2) = 4$, $f(x_0,x_1,x_2,x_3) = 1$

Substituting these values in (1)

$$f(x) = 2 + (x-0)\cdot 1 + (x-0)(x-1)\cdot 4 + (x-0)(x-1)(x-2)\cdot 1$$

$$= 2 + x + 4x(x-1) + x(x-1)(x-2)$$

$$\boxed{f(x) = x^3 + x^2 - x + 2}$$

$$f(3) = 27 + 9 - 3 + 2 = 35$$

5. Use Newton's divided difference formula, to fit a polynomial to the data

x	:	-1	0	2	3
y	:	-8	3	1	12

and hence find y when $x=1$.

Solution :-

The divided difference table for the given data is as follows.

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
$x_0 \to -1$	$-8 \to f(x_0)$	$f(x_0,x_1)$: $\frac{3+8}{0+1}=11$	$f(x_0,x_1,x_2)$: $\frac{-1-11}{2+1}=-4$	$f(x_0,x_1,x_2,x_3)$: $\frac{4+4}{3+1}=2$
$x_1 \to 0$	3	$\frac{1-3}{2-0}=-1$	$\frac{11+1}{3-0}=4$	
$x_2 \to 2$	1	$\frac{12-1}{3-2}=11$		
$x_3 \to 3$	12			

$\therefore$ By Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$

$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \dots$$

$$f(x) = -8 + (x+1)11 + (x+1)x(-4) + (x+1)x(x-2)2$$

$$= -8 + 11x + 11 - 4x^2 - 4x + 2x^3 - 2x^2 - 4x$$

$$\therefore\ y = 2x^3 - 6x^2 + 3x + 3.$$

$$\Rightarrow y(1) = 2 - \cancel{6} + \cancel{3} + \cancel{3} = 2$$

6. Using Newton's divided difference formula, find the value of $f(8)$ and $f(5)$ given the following data.

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

<u>Solution</u>:

Form the divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
$x_0 \to 4$	48 $f(x_0)$	$f(x_0,x_1)$: $\frac{100-48}{5-4}=52$	$f(x_0,x_1,x_2)$: $\frac{97-52}{7-4}=15$	$f(x_0,x_1,x_2,x_3)$: $\frac{21-15}{10-4}=1$	
$x_1 \to 5$	100	$\frac{294-100}{7-5}=97$	$\frac{202-97}{10-5}=21$	$\frac{27-21}{11-5}=1$	0
$x_2 \to 7$	294	$\frac{900-294}{10-7}=202$	$\frac{310-202}{11-7}=27$	$\frac{33-27}{13-7}=1$	0
$x_3 \to 10$	900	$\frac{1210-900}{11-10}=310$	$\frac{409-310}{13-10}=33$		
$x_4 \to 11$	1210	$\frac{2028-1210}{13-11}=409$			
$x_5 \to 13$	2028				

By Newton's divided difference interpolation formula.

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \cdots$$

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + (x-4)(x-5)(x-7)$$

when $x = 8$,

$$f(8) = 48 + 208 + 180 + 12$$

$$\boxed{f(8) = 448}$$

when $x = 15$,

$f(15) = 48 + 572 + 1650 + 880$

$\Rightarrow f(15) = 3150$.

Homework problems:

1. Using divided difference, find $f(8)$ from the following :

x :	3	7	9	10
$f(x)$:	168	120	72	63

Ans: $f(8) = 93$.

2. Using Newton's divided difference formula find $f(x)$ and $f(6)$ from the following data :

x :	1	2	7	8
$f(x)$:	1	5	5	4

Ans:

$$f(x) = \frac{1}{42}\left[3x^3 - 58x^2 + 321x - 224\right]$$

$$f(6) = 6.2381.$$